

Elliptic Curves $x^3 + y^3 = k$ of High Rank

Noam D. Elkies¹ and Nicholas F. Rogers²

¹ Department of Mathematics, Harvard University, Cambridge, MA 02140
`elkies@math.harvard.edu`

Supported in part by NSF grant DMS-0200687

² Department of Mathematics, Harvard University, Cambridge, MA 02140
`nfrogers@math.harvard.edu`

Abstract. We use rational parametrizations of certain cubic surfaces and an explicit formula for descent via 3-isogeny to construct the first examples of elliptic curves $E_k : x^3 + y^3 = k$ of ranks 8, 9, 10, and 11 over \mathbb{Q} . As a corollary we produce examples of elliptic curves over \mathbb{Q} with a rational 3-torsion point and rank as high as 11. We also discuss the problem of finding the minimal curve E_k of a given rank, in the sense of both $|k|$ and the conductor of E_k , and we give some new results in this direction. We include descriptions of the relevant algorithms and heuristics, as well as numerical data.

1 Introduction

In the fundamental Diophantine problem of finding rational points on an elliptic curve E , one is naturally led to ask which abelian groups can occur as the group of rational points $E(\mathbb{Q})$. Mordell’s theorem guarantees that $E(\mathbb{Q})$ is finitely generated, so we have

$$E(\mathbb{Q}) = E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r,$$

where r is the rank of E . Mazur’s well-known work [Ma] completely classifies the possibilities for $E(\mathbb{Q})_{\text{tors}}$, but the behavior of the rank remains mysterious. Part of the “folklore” is the conjecture that there exist elliptic curves with arbitrarily large rank over \mathbb{Q} . But large rank examples are rare, and the record to date is 24 [MM]. One might further ask about the distribution of ranks in families of twists, or with prescribed Galois structure on the torsion subgroup; there is some evidence to suggest that conditions of this sort do not impose an upper bound on the rank.

A classical question in number theory is to describe the numbers k that can be written as the sum of two rational cubes. This leads one to study the elliptic curves

$$E_k : x^3 + y^3 = k$$

for $k \in \mathbb{Q}^*$. Clearly E_k and $E_{k'}$ are isomorphic if k/k' is a cube, so we can and will restrict our attention to positive cubefree integers k . A Weierstrass equation for E_k is given by $Y^2 = X^3 - 432k^2$, where

$$X = \frac{12k}{y+x}, \quad Y = 36k \frac{y-x}{y+x}.$$

As long as $k > 2$, the group $E_k(\mathbb{Q})_{\text{tors}}$ is trivial, so E_k has a nontrivial rational point if and only if its rank is positive. The distribution of ranks in this family is not well understood. Zagier and Kramarz [ZK] used numerical evidence for $k \leq 70000$ to conjecture that a positive proportion of the curves E_k have rank at least 2; however, more recent computations by Mark Watkins [Wa] suggest that, in fact, a curve E_k has rank 0 or 1 with probability 1. Still, the following conjecture is widely believed:

Conjecture 1 *There exist elliptic curves E_k with arbitrarily large rank over \mathbb{Q} .*

A proof of this conjecture seems beyond the reach of current techniques. So for now we content ourselves with constructing high-rank examples within this family (thereby adding to the body of supporting evidence), and gathering more data on the distribution of ranks so as to be able to formulate more precise conjectures. The main results of this paper are examples of curves E_k with rank r for each $r \leq 11$. For $r = 8, 9, 10, 11$ these are the first curves known of those ranks; for $r = 6, 7$ our curves have k smaller than previous records, and are proved minimal assuming some standard conjectures. For $r \leq 5$ we recover previously known k , and prove unconditionally that they are minimal.

Throughout, we make use of the fact that the curves E_k are 3-isogenous to the curves

$$E'_k : uv(u + v) = k$$

or, in Weierstrass form, $E'_k : V^2 = U^3 + 16k^2$, where

$$U = \frac{4k}{v}, \quad V = \frac{8ku + 4kv}{v}.$$

The isogeny is given by:

$$\phi : E_k \rightarrow E'_k, \quad (x, y) \mapsto (u, v) = \left(\frac{y^2}{x}, -\frac{k}{xy}\right).$$

The dual isogeny, with respect to the Weierstrass equations for E_k and E'_k , is

$$\hat{\phi} : E'_k \rightarrow E_k, \quad (U, V) \mapsto (X, Y) = \left(\frac{U^3 + 64k^2}{U^2}, \frac{V(U^3 - 128k^2)}{U^3}\right).$$

Applying Tate's Algorithm [Ta] to the curves E'_k , we find that a minimal Weierstrass form for E'_k is given by

$$Z^2 = W^3 + \frac{k^2}{4} \quad (W, Z) = \left(\frac{U}{4}, \frac{V}{8}\right)$$

in the case that k is even, and

$$Z^2 + Z = W^3 + \frac{k^2 - 1}{4} \quad (W, Z) = \left(\frac{U}{4}, \frac{V - 4}{8}\right)$$

in the case that k is odd. The primes of bad reduction for E'_k are the primes dividing k and the prime 3. For a prime factor p of k , the Kodaira type at p is

IV* if $p^2 \mid k$ and IV if $p \parallel k$. If $3 \nmid k$, the Kodaira type at 3 is III if $k \equiv \pm 2 \pmod{9}$, and II otherwise. It follows that the conductor of E'_k is given by the formula

$$N(E'_k) = \prod_{p \mid 3k} p^{2+\beta_p}$$

where $\beta_p = 0$ if $p \neq 3$, and $\beta_3 = 0, 1$, or 3 for $k \equiv \pm 2 \pmod{9}$, $k \equiv \pm 1, \pm 4 \pmod{9}$, or $3 \mid k$ respectively.

The curves E'_k have the rational 3-torsion points $(U, V) = (0, \pm 4k)$. Since the rank is an isogeny invariant, we produce as a corollary to our work examples of elliptic curves E'_k with a rational 3-torsion point and rank as high as 11. Curiously, there are no other known elliptic curves over \mathbb{Q} with a rational 3-torsion point and rank greater than 8 [Du].

In section 2 we describe the geometric underpinnings of our search technique, which heavily uses rational parametrizations of various cubic surfaces, the points of which correspond to pairs of (usually independent) points on the curves E_k . Section 3 gives a formula for an upper bound on the rank of E_k , using descent via 3-isogeny. Section 4 describes some of the specific algorithms we used to produce examples of E_k with high rank. Finally, we give our numerical results in Section 5.

2 Cubic Surfaces

The most naïve approach to constructing curves E_k of high rank is to enumerate small points on the curves E_k , which can be accomplished via the simple observation that a point on some curve E_k corresponds to a pair of whole numbers (x, y) so that the cubefree part of $x^3 + y^3$ (that is, the unique cubefree integer s such that $(x^3 + y^3)/s$ is a perfect cube) is k . The second author used essentially this approach to find the first known E_k of rank 7. By incorporating some more sophisticated techniques, such as 3-descent (see below), this approach could yield curves with rank as high as 8. The weakness of this method is that the number of such points up to height H grows as H^2 , most of which lie on curves of rank 1 and waste our time and/or memory.

We can reduce this H^2 to $H^{1+\epsilon}$ by considering only curves E_k together with a pair of points, which correspond to points on the cubic surface

$$S_1 : w^3 + x^3 = y^3 + z^3$$

other than the trivial points on the lines $w + x = y + z = 0$, $w + y = x + z = 0$, $w + z = x + y = 0$. (This pairs-of-points idea is also used in [EW] to produce elliptic curves with high rank and smallest conductor known.) The cubic surface whose points correspond to pairs of points on the isogenous curves E'_k ,

$$S_2 : wx(w + x) = yz(y + z),$$

and the “mixed” cubic surface

$$S_3 : wx(w + x) = y^3 + z^3,$$

are also fruitful. Each of these cubic surfaces is smooth, and thus rational in the sense that it is birational to \mathbb{P}^2 over $\overline{\mathbb{Q}}$; in fact, each has a rational parametrization defined over \mathbb{Q} .

A parametrization of S_1 was found by the first author [El], and a parametrization of S_2 follows fairly quickly: there is an obvious isomorphism between S_1 and S_2 , defined *a priori* over $\mathbb{Q}(\sqrt{-3})$, but which actually descends to \mathbb{Q} . To parametrize S_3 , we used the following more general approach, provided by Izzet Coskun [Co].

Let S be a cubic surface defined over \mathbb{Q} , and suppose L_1 and L_2 are disjoint lines on S , with L_3 a third line meeting both. Then there is a 3-dimensional space of quadrics that vanish on this set of lines. Use a basis of this space to map S into \mathbb{P}^2 ; the inverse map will be a parametrization of S . The parametrization so obtained is defined over \mathbb{Q} if all of the L_i are rational, or if L_3 is rational and L_1, L_2 are Galois conjugate. This construction realizes S as \mathbb{P}^2 blown up at six points; the six blown-down lines are the six lines (other than L_i) that meet exactly two of the three lines L_1, L_2, L_3 .

On both S_1 and S_3 , the relative paucity of lines defined over \mathbb{Q} means that, up to automorphism of the surface, there is only one choice for the configuration L_1, L_2, L_3 . Thus for S_1 the parametrization obtained with this technique,

$$\begin{aligned} (w : x : y : z) = & (t^3 - 2t^2s - 2tsr + ts^2 + r^2s - r^3 - rs^2 \\ & : -t^3 - 2t^2s + ts^2 - 2tsr + 2rs^2 - s^3 + r^3 - 2r^2s \\ & : 2t^3 + 3t^2r - 2t^2s - 2tsr + 3tr^2 + ts^2 + 2rs^2 - s^3 + r^3 - 2r^2s \\ & : -2t^3 + t^2s - 3t^2r - 3tr^2 + 4tsr - 2ts^2 - r^3 + r^2s - rs^2), \end{aligned}$$

is equivalent to the one in [El], in the sense that one can be obtained from the other by composing an automorphism of S_1 with a projective linear transformation of \mathbb{P}^2 . For S_3 we obtain the parametrization

$$(w : x : y : z) = (r^3 - s^3 : s^3 + t^3 : r^2s - s^2t + t^2r : r^2t - s^2r - t^2s).$$

In order to compute the k for which a particular point on S_3 corresponds to a pair of points on E'_k and E_k , we must find the cubefree part of $wx(w+x)$, which we do by factoring this number. It is therefore useful that $wx(w+x)$, which is a polynomial of degree 9 in r, s , and t , decomposes as a product of three linear and three quadratic factors. By contrast, the factorization of $w^3 + x^3$ in the parametrization of S_1 is as a product of one linear, two quadratic, and one quartic factor; the difficulty of factoring the value of this quartic at (r, s, t) severely limits the usefulness of the S_1 parametrization.

On S_2 , there are, up to automorphism, seven different configurations of lines L_1, L_2, L_3 . Four of them lead to parametrizations where $wx(w+x)$ has five linear and two quadratic factors; the parametrization of S_2 mentioned above is of this type. The other three lead to factorizations into three linear and three quadratic polynomials. There is not a significant computational advantage for one factorization over another, but we do mention here a rather elegant parametrization of S_2 obtained in this way:

$$(w : x : y : z) = (-r^2s + s^2t : r^2s - rt^2 : -r^2t + st^2 : rs^2 - st^2).$$

3 Descent via 3-isogeny

A powerful tool for obtaining upper bounds for the ranks of the curves E_k is descent, since these curves admit the aforementioned 3-isogeny with E'_k . An analysis of the descent for these curves first appeared in [Se]. What follows is essentially a simplification of the formula given there.

Let $k = \prod p_j^{\epsilon_j}$, where $\epsilon_j = 1$ or 2 , and let q_i be the primes dividing k with $q_i \equiv 1 \pmod{3}$. Now define a matrix A over \mathbb{F}_3 by:

$$\left(\frac{p_j^{\epsilon_j}}{q_i}\right)_3 = \rho^{A_{ij}}$$

if $p_j \neq q_i$, and $A_{ij} = -\sum_{\ell: p_\ell \neq q_i} A_{i\ell}$ if $p_j = q_i$ (equivalently, the rows of A sum to zero). Here $\left(\frac{\cdot}{q}\right)_3$ denotes the cubic residue symbol mod q ; note that for each $q \equiv 1 \pmod{3}$, there are two choices for this cubic residue symbol, but they lead to proportional rows A_i . If $k \equiv \pm 1$ or $0 \pmod{9}$, add an additional row corresponding to cubic characters mod 9 for p_j ; if $9|k$, the entry corresponding to $p_j = q_i = 3$ is defined as in general when $p_j = q_i$.

If we let $\phi : E_k \rightarrow E'_k$ and $\phi' : E'_k \rightarrow E_k$ denote the relevant 3-isogenies, then the row and column null spaces of A correspond, respectively, to the ϕ - and ϕ' -Selmer groups of E_k/\mathbb{Q} and E'_k/\mathbb{Q} . We can conclude that, after taking the 3-torsion of E'_k into account,

$$\text{rank}(E_k) \leq \#\text{rows} + \#\text{columns} - 2 \cdot \text{rank}(A) - 1.$$

4 Computational Techniques

An application of the explicit formula for descent via 3-isogeny is another technique for searching for curves of large rank, which tends to be more effective in finding the minimal k such that E_k has a given rank (it was actually this technique that produced the current rank 9 record). The idea is to enumerate all possible k less than some given upper bound which have a sufficiently high 3-Selmer bound, and then to search for points on these curves. To do this, we recursively build up products of primes, and at each stage compute the portion of the matrix A corresponding to the primes chosen so far. Of course, the diagonal entries remain in some doubt, since they depend on the whole row of A ; still, at each stage a lower bound for the rank of the matrix A can be computed, and used to give a lower bound on the number of primes still needed. In this way one can vastly reduce the search space. A similar approach can be used to enumerate candidate curves whose conductor is smaller than a given bound, with the formula for the conductor given in the Introduction.

It is also important to have a way to guess which curves are the most promising before committing to a lengthy point search. The most important tool in this regard is provided by a heuristic argument suggesting that high rank curves

should have many points on their reductions modulo p , or more specifically,

$$\prod_{p \leq x} \frac{\#E(\mathbb{F}_p)}{p+1} \sim (\log x)^r$$

where r is the rank of E . This formula was conjectured by Birch and Swinnerton-Dyer in [BSD], and the idea of using it to find elliptic curves of high rank is due to Mestre [Me].

Note that for the curves E_k it is only useful to consider primes $p \equiv 1 \pmod{3}$, since E_k is supersingular mod p when $p \equiv 2 \pmod{3}$. Furthermore, computing $\#E_k(\mathbb{F}_p)$ is quite fast: modulo each prime $p \equiv 1 \pmod{3}$, there are only three isomorphism classes of E_k , corresponding to the three cubic residue symbols mod p . We compute the a_p 's for each of these isomorphism classes once for all, and then to find the a_p for a given curve E_k , we need only compute the cubic residue symbol of $k \pmod{p}$.

In the end, though, we must still search for points on curves we suspect of having high rank. But here, too, there are improvements over the most naïve approach. As noted above, points on E_k correspond to pairs of whole numbers (x, y) such that $k = d^{-3}(x^3 + y^3)$. Of course, we may further assume that x and y are relatively prime. It follows that $\gcd(x + y, x^2 - xy + y^2)$ is either 1 or 3, whence $x + y$ must be a factor of $3k$ times a perfect cube. For each $x + y$ and d , we must simply decide if the resulting quadratic equation has a rational solution. Furthermore, we can use local conditions to reduce the number of possible $x + y$ we must consider (this is closely related to the descent described in the last section). A similar approach works for a point search on $xy(x + y) = k$.

5 Results

Here we list the minimal known k such that E_k has rank r for each rank $r \leq 11$, as well as the minimal known conductor of a curve E_k of rank r with $r \leq 8$. We include notes where relevant, and r independent points for each of the new record curves. The points are listed on the minimal Weierstrass equation for E'_k , as described in the Introduction. To transfer points back to the curve E_k one may use the dual isogeny $\hat{\phi}$ described there.

The following records for ranks up to 5 are known to be minimal (the proof for ranks 4 and 5 seems to be new); for ranks 6 and 7, they are minimal provided the weak Birch and Swinnerton-Dyer Conjecture and the Generalized Riemann Hypothesis are true for all $L(E_{k'}, s)$ with $k' < k$. The records for ranks 8 through 10 are likely to be minimal. In each case, we use the approach described in the last section to enumerate all of the smaller k with a sufficiently large 3-Selmer group. For each of them we compute a partial product of $L(E_k, 1)$ over the first 1000 or so primes; a large partial product should correspond to high rank. In each case of rank 8 through 10, the record k significantly distinguished itself from all smaller k . Note that for large r our record value of k tends to have considerably more prime factors congruent to 1 mod 3 than to 2 mod 3.

rank	k (minimal known with E_k of given rank)
0	1
1	$6 = 2 \cdot 3$
2	19 (prime)
3	$657 = 3 \cdot 3 \cdot 73$
4	$21691 = 109 \cdot 199$
5	$489489 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 163$
6	$9902523 = 3 \cdot 73 \cdot 103 \cdot 439$
7	$1144421889 = 3 \cdot 13 \cdot 19 \cdot 41 \cdot 139 \cdot 271$
8	$1683200989470 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 29 \cdot 41 \cdot 47 \cdot 59$
9	$349043376293530 = 2 \cdot 5 \cdot 37 \cdot 41 \cdot 53 \cdot 73 \cdot 1231 \cdot 4831$
10	$137006962414679910 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 23 \cdot 31 \cdot 37 \cdot 43 \cdot 83 \cdot 109 \cdot 151 \cdot 421$
11	$13293998056584952174157235$ $= 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 43 \cdot 59 \cdot 61 \cdot 73 \cdot 79 \cdot 103 \cdot 109 \cdot 157 \cdot 457$

The k in the following chart are known to correspond to curves E_k of minimal conductor for $k \leq 3$. For ranks 4, 5, and 6, they are minimal provided the weak Birch and Swinnerton-Dyer Conjecture and the Generalized Riemann Hypothesis are true for all $L(E_{k'}, s)$ with $k' < k$. The records for ranks 7 and 8 are likely to be minimal; as above, Mestre's heuristic distinguishes them markedly from all curves of smaller conductor. It is striking that for all ranks less than 8, the k corresponding to minimal conductor are squarefree away from 3. Since divisibility by p^2 as opposed to p in k does not alter the value of the conductor of E_k , one might expect that for high rank, the k corresponding to the curve of minimal conductor would have many square factors.

rank	k (E_k of given rank and minimal known conductor)
0	1
1	$9 = 3^2$
2	19 (prime)
3	$657 = 3 \cdot 3 \cdot 73$
4	$34706 = 2 \cdot 7 \cdot 37 \cdot 67$
5	$763002 = 2 \cdot 3^2 \cdot 19 \cdot 23 \cdot 97$
6	$24565833 = 3^2 \cdot 17 \cdot 307 \cdot 523$
7	$1144421889 = 3 \cdot 13 \cdot 19 \cdot 41 \cdot 139 \cdot 271$
8	$23381862574950 = 2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 19 \cdot 23 \cdot 31 \cdot 83 \cdot 4201$

We conclude by listing independent points on the record curves E'_k for ranks 6 through 11, all of which were newly found using methods described here. A program that implements LLL reduction on the lattice of points of E_k , provided by Randall Rathbun [Ra], was used to reduce their heights where possible.

$$k = 9902523, r = 6$$

6 independent points on the Weierstrass minimal curve for E'_k :

$$y^2 + y = x^3 + 24514990441382$$

(100092, 32051170), (−6798, 4919434), (−22338, 3656314),
(43672, 10383069), (−11988, 4774114), (126720, 45380386).

$$k = 24565833, r = 6$$

6 independent points on the Weierstrass minimal curve for E'_k :

$$y^2 + y = x^3 + 150870037745972$$

(−37656, 9872932), (86292, 28167835), (187270, 81966083),
(−32058, 10859260), (−39798, 9372019), (236572, 115719195).

$$k = 1144421889, r = 7$$

7 independent points on the Weierstrass minimal curve for E'_k :

$$y^2 + y = x^3 + 327425365005582080$$

(267748, 588744383), (1235988, 1488490048), (−333330, 538877944),
(−648774, 233133847), (−422760, 501863680), (5104008, 11545190143),
(−688974, 19483912).

$$k = 1683200989470, r = 8$$

8 independent points on the Weierstrass minimal curve for E'_k :

$$y^2 = x^3 + 708291392738196762720225$$

(−88860785, 81396479060), (−87348261, 204569627262),
(−63256830, 674665720365), (−40588401, 800890328532),
(101707060, 1326794024465), (−35705670, 814107128835),
(−44793980, 786391914385), (8308684440, 757353550270065).

In fact, the curve E'_k for $k = 1683200989470$ has the remarkable property that the Diophantine equation $xy(x+y) = k$ actually has 8 *integral* solutions, namely (11, 391170), (533, 55930), (770, 46371), (1003, 40467), (2639, 23970), (6970, 12441), (7293, 1197), (8555, 10387). These 8 solutions, considered as rational points on E'_k , are independent.

$$k = 23381862574950, r = 8$$

8 independent points on the Weierstrass minimal curve for E'_k :

$$y^2 = x^3 + 136677874368461861091875625$$

(1826174700, 78910161016275), (794409100, 25259021976275),
 (-259483950, 10918164759975), (499986216, 16176140969139),
 (-503804925, 2966885463150), (798185799, 25400836633032),
 (165873591, 11884516327764), (215137494, 12109307517153).

$$k = 349043376293530, r = 9$$

9 independent points on the Weierstrass minimal curve for E'_k :

$$y^2 = x^3 + 30457819633596695100179965225 :$$

(-734843410, 173381106196815), (5130038900, 406775844709485),
 (-2676929565, 106184137901590), (690947990, 175464197227935),
 (291207945620, 157146639625792365), (25120488440, 3985280944128435),
 (-872639080, 172607374924365), (2918890200, 235215923409485),
 (-102315705, 174518619468760).

$$k = 137006962414679910, r = 10$$

10 independent points on the Weierstrass minimal curve for E'_k :

$$y^2 = x^3 + 4692726937524378378756566939402025 :$$

(-135797482140, 46781315964225555), (-150436201545, 35891470127810220),
 (-42200591214, 67952721291406041), (2327642247924, 3551854243978575507),
 (5504535148140, 12914782107290941395), (140506152430, 86409469562070095),
 (397507563420, 259814927561209005), (7162660587075, 19169656506442936830),
 (73148794740, 71303068454026605), (-102758626586, 60063833881519937).

$$k = 13293998056584952174157235, r = 11$$

11 independent points on the Weierstrass minimal curve for E'_k :

$$y^2 + y = x^3 + 44182596082121121317135170025680399046545625711306 : \\ (-30156002278649820, 4093799681127459731025817), \\ (11364087102067560, 6756491872572362690626342), \\ (-20835788771691894, 5927660006237675713476241), \\ (1134264920569989390, 1208031685828825118221478017), \\ (8907565209691176834, 26585114133655761890666064910), \\ (111849199886121334, 37992674604901443769570910), \\ (11724873521668020, 6767159346634715672034457), \\ (-138658831412368575/4, 12719819443574268333325811/8), \\ (165971060901522240, 67941788876402816577138982), \\ (994768217796990, 6647073075327662243966017), \\ (532896351059436225/16, 576457310785324883248677823/64).$$

References

- BSD. Birch, B.J., and Swinnerton-Dyer, H.P.F.: Notes on Elliptic Curves II. *J. reine angew. Math.* **218** (1965) 79–108.
- Co. Coskun, I., personal communication.
- Du. Dujella, A.: High rank elliptic curves with prescribed torsion. <http://www.math.hr/~duje/tors/tors.html> (2000–2003).
- El. Elkies, N.D.: Complete Cubic Parametrization of the Fermat Cubic Surface. 2001 (<http://math.harvard.edu/~elkies/4cubes.html>).
- EW. Elkies, N.D., and Watkins, M.: Elliptic curves of large rank and small conductor. Preprint, 2003 (accepted for ANTS-6).
- Ma. Mazur, B.: Modular curves and the Eisenstein ideal. *IHES Publ. Math.* **47** (1977) 33–186.
- Me. Mestre, J.-F.: Courbes elliptiques de rang ≥ 11 sur $\mathbb{Q}(t)$. (French) [Elliptic curves with rank ≥ 11 over $\mathbb{Q}(t)$.] *C. R. Acad. Sci. Paris Sér. I Math.* **313** (1991) 139–142.
- MM. Martin, R., and McMillen, W.: An elliptic curve over \mathbb{Q} with rank at least 24. Number Theory Listserver, May 2000. Electronically available from listserv.nodak.edu/archives/nmbrthry.html and directly from www.math.harvard.edu/~elkies/rk24_1.html
- Ra. Rathbun, R., personal communication.
- Se. Selmer, E.S.: The Diophantine Equation $ax^3 + by^3 + cz^3 = 0$. *Acta. Math.* **85** (1951) 203–362.
- Si. Silverman, J.H.: *The Arithmetic of Elliptic Curves*. New York: Springer, 1986.
- Ta. Tate, J.: Algorithm for determining the type of a singular fiber in an elliptic pencil. In *Modular Functions of One Variable IV*, Lect. Notes in Math. 476, B.J. Birch and W. Kuyk, eds. Berlin: Springer-Verlag, 1975, 33–52.
- Wa. Watkins, M.: Rank Distribution in a Family of Cubic Twists. Preprint, 2003 (<http://www.math.psu.edu/watkins/papers/zk.ps>).
- ZK. Zagier, D., and Kramarz, G.: Numerical investigations related to the L -series of certain elliptic curves. *J. Indian Math. Soc.* **52** (1987), 51–69.